nated area and the duration of the effect at the receiving point and Δt is the interval between successive peaks at that point.

Table 8 shows that Δt is several seconds, whereas δt is less than 1 sec. This is strictly true, of course, only if the body is exposed to plane waves from one direction only; transmitters at several points S_1, S_2, S_3 (Fig. 15) give rise to several scattered waves at point E, so that the effect lasts for a substantially longer time.

Finally, σ exceeds σ_0 only for bodies of small size if there is no magnetic field; for example, $\sigma_{max} \approx 0.005$ m^2 for $\lambda = 30$ m, $z = 300$ km, and $R_0 = 0.5$ m, so that $\sigma_{max}/\sigma_0 \approx 7$ (ionization produces much the greater scattering), whereas $\sigma_{max} \approx 0.02$ m^2 and $\sigma_{max}/\sigma_0 \approx 0.5$ for $R_0 = 1$ m, so that the sphere scatters more than the track. Further, σ varies as $1/\epsilon$ if the dielectric constant of the plasma ϵ alters; therefore, the effect should be substantially greater if the body lies in a region in which ϵ approaches zero. However, this last case demands special examination; no precise deductions of the behavior of σ as $\epsilon \to 0$ are possible without a detailed analysis.

—*Received August 4, 1961*

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Reviewer's Comment

The results of a previous contribution* are used to present numerical results (based on computer calculations) on the scattering cross section of a wake induced by a spherical satellite vehicle traversing typical regions of the ionosphere. The influence of a magnetic field is specifically included. Effects of height, frequency, ion temperature, vehicle velocity, and geometry (direction of magnetic field, vehicle velocity, and earth stations) are illustrated by the calculations.

The contribution is noteworthy in that at least three significant conclusions can be drawn from the numerical results. These are:

1) The scattering cross section is a maximum at the mirror image reflecting points only when the vehicle velocity is parallel to the magnetic field.

2) When the vehicle velocity is not parallel to the magnetic field, major maxima occur at angles off the specular direction, the deviation depending upon vehicle velocity and scattering angle of the wave.

3) A series of maxima and minima in the scattering cross section occurs. These form a symmetrical sequence if the vehicle moves parallel to the magnetic field, but are notably unsymmetrical (in magnitude) at other directions of the vehicle motion relative to the magnetic field.

An attempt to interpret high frequency radio reflections observed from orbiting satellites† in terms of this theory and radio observations designed to verify these predictions would form a worthy contribution in the future.

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* Pitaevskii, L. P., Geomagnetizm i Aeronomiia (Geomagnetism and Aeronomy) **1**, no. 2, 194–208 (translated on pp. 994–1000 of this issue). Unfortunately the notation used in the forementioned and the present article are somewhat different.

† For instance: Krauss, J. D., et al., Proc. Inst. Radio Engrs. **48**, 672–678, 1913–1914 (1960).

APRIL 1963 AIAA JOURNAL VOL. 1, NO. 4

Analog Computer Solution of the Problem of Accumulation of Perturbations

G. V. Savinov

PROBLEMS involving the dynamic accuracy of automatic control systems are gaining importance steadily because of the more exacting engineering requirements imposed on such systems. In this context, particular interest centers on a buildup of perturbations, a problem that was posed and solved for linear systems by Bulgakov and developed further in subsequent work by Roitenberg, Kuzovkov, and others.[1-4]

The present paper deals with a dynamical system subjected over a finite time interval to the action of perturbing forces bounded in absolute value. We consider the determination of maximum deviation of any given coordinate of the dynamical system involved, the peak value being accumulated to some preassigned instant of time under the most unfavorable perturbation conditions.

The solution of this problem requires integration of an auxiliary system of differential equations, known as an adjoint system of equations—a task suited to analog computers.

The procedure followed in using analog computers to solve the problem of cumulative perturbations in linear and nonlinear systems, as well as the "hit" problem, which is strikingly similar in character to the problem of accumulated perturbations, will be outlined.

Translated from Vestnik Moskovskogo Universiteta, Seriia I: Matematika i Mekhanika (Bulletin of Moscow University, Series I: Mathematics and Mechanics), no. 3, 62–76 (1961). Translated by Faraday Translations, New York.

Solution by Analog Computer of the Problem of a Buildup of Perturbations in Linear Systems

Consider a dynamical system whose motion is described by the following linear differential equations with variable coefficients:

$$x_i = \sum_{k=1}^{n} a_{ik}(t)x_k + f_i(t) \qquad i = 1, \ldots, n \tag{1}$$

Here, $x_1, \ldots, x_n$ are the system coordinates; $a_{ik}(t)$ the time-varying system parameters; $f_i(t)$ the external perturbations or inputs acting on the system, and satisfying the constraint

$$|f_i(t)| \leqslant K_i$$

The problem reduces to a determination of the magnitude of $|x_k(T)|_{\max}$, i.e., the maximum cumulative deviation of the kth coordinate of the dynamical system by some predetermined instant of time T under worst-case perturbation conditions when the greatest deviation in the kth coordinate is brought about.

In addition to the system of equations (1), we consider also an auxiliary system, again known as the adjoint system, of differential equations:

$$\dot{p}_i = -\sum_{k=1}^{n} a_{ki}(t)p_k \qquad i = 1, \ldots, n \tag{2}$$

It is seen readily that the matrix of the coefficients of the adjoint system is obtained by changing the sign and transposing the matrix of the original system.

On multiplying both sides of each equation in system (1) by p_i and in system (2) by the factor x_i, we find, after adding all the equations in system (1) to the equations in system (2):

$$\frac{d}{dt}\sum_{i=1}^{n} x_i p_i = \sum_{i=1}^{n} p_i f_i$$

Integrating the expression so obtained from 0 to t, we find

$$\sum_{i=1}^{n} x_i(t)p_i(t) = \sum_{i=1}^{n} x_i(0)p_i(0) + \int_0^t \sum_{i=1}^{n} p_i(\tau)f_i(\tau)d\tau \tag{3}$$

Let $x_i(0) = 0$. Then, for time $t = T$, we have

$$\sum_{i=1}^{n} x_i(T)p_i(T) = \int_0^T \sum_{i=1}^{n} p_i(\tau)f_i(\tau)d\tau \tag{4}$$

The functions $p_i(t)$ figuring in these formulas are not completely defined, since no initial conditions have been specified as yet for p_i.

Let us now assign initial conditions for p_i such that

$$\begin{array}{lll} p_i(T) = 1 & \text{for} & i = k \\ p_i(T) = 0 & \text{for} & i \neq k \end{array} \tag{5}$$

Then, from Eq. (4), we have

$$x_k(T) = \int_0^T \sum_{i=1}^{n} p_i(\tau)f_i(\tau)d\tau \tag{6}$$

Equation (6) provides the relationship between the magnitude of the kth coordinate of the system and the magnitudes of the external disturbances. This makes it directly evident that the peak accumulated deviation of the kth coordinate by some prespecified instant of time T is governed by the following law regarding the variation of the external perturbations:

$$f_i(t) = K_i \operatorname{sign} p_i(t) \qquad i = 1, \ldots, n \tag{7}$$

whereupon the value of the peak cumulative deviation in the kth coordinate of the system is

$$|x_k(t)|_{\max} = \sum_{i=1}^{n} K_i \int_0^T |p_i(\tau)|d\tau$$

From Eqs. (6) and (7), we see that the worst case of external perturbations in which the deviation of the kth coordinate of the system reaches its peak value by time T is obtained when the external perturbing forces $f_i(t)$ assume their maximum possible values, and the sign of these perturbations must be the same as the sign of functions $p_i(t)$.

As stated earlier, functions $p_i(t)$ are defined by the adjoint system (2), which is integrable at the initial values of the variables satisfying conditions (5).

Functions $p_i(t)$, required to compute the maximum cumulative deviation of coordinate $x_k(T)$, may be obtained by integrating the system of differential equations (2) with the aid of analog computers.

It is known generally that analog computers can simulate systems described by a system of differential equations with predetermined initial conditions (5).

Since only finite values of variables $p_i(T)$ are known in the problem under discussion, it is first necessary to find the initial values for p_i. The latter may be found by integrating system (2) "backwards," i.e., by introducing a new independent variable τ defined by the expression $t = T - \tau$. After this change of independent variable, the system of equations (2) transforms to the form

$$-\frac{d}{d\tau} p_i(\tau) = \sum_{k=1}^{n} a_{ki}(T - \tau)p_k(\tau) \qquad i = 1, \ldots, n \tag{8}$$

The system of equations (8) can be integrated in turn by means of analog computers, with the following initial conditions [see Eq. (5)]:

$$\begin{array}{l} p_i(T - \tau)|_{\tau=0} = p_i(T) = 0 \quad \text{for} \quad i \neq k \\ p_k(T - \tau)|_{\tau=0} = 1 \end{array} \tag{9}$$

After a time interval equal to T has elapsed, the values of all the variables must be fixed. Since $p_i(T - \tau)_{\tau=T} = p_i(0)$, the fixed values of the variables in system of equations (8) are, in fact, the initial values we need for the variables in the adjoint system of equations (2). Although the initial system (1) i.e., both (1) and the adjoint system of equations (2), are systems with constant coefficients, system (8) differs from the adjoint system solely by the signs accompanying all the derivatives, and it is particularly easy to carry out the integration of system (8) in that case.

On the basis of the discussion up to this point, we suggest the following procedure for utilizing analog computers to find the maximum cumulative deviation in the kth coordinate of a linear system when the disturbances acting on the system are bounded with respect to absolute value.

A System of equations (8), which is integrated over the interval $0 \leqslant \tau \leqslant T$ at the initial conditions specified in (9), is simulated by analog techniques. A set of values of $p(0)$, is determined as a result of the integration.

B The adjoint system (2), which is integrated over the range $0 \leqslant t \leqslant T$ at the initial values found previously, is simulated. In the process of integrating the adjoint system (2), the aid of additional functional units makes available values of b_i, where

$$b_i = \int_0^T |p_i(\tau)|d\tau \qquad i = 1, \ldots, n$$

C The value of the maximum cumulative deviation of the system kth coordinate during time T is calculated from the formula

$$|x_k(T)|_{\max} = \sum_{i=1}^{n} K_i b_i$$

As an example, consider the dynamical system described by

$$\begin{array}{l} \dot{x}_1 = 0.3x_2 + f_1(t) \\ \dot{x}_2 = -0.3x_1 - 0.04x_2 \qquad x_1(0) = x_2(0) = 0 \end{array} \tag{10}$$

Let $T = 15$ sec and $|f_1(T)| \leqslant K_1 = 0.1$. It is necessary to determine the maximum accumulated deviation with respect to coordinate x_1, that is, to calculate the value of $|x_1(T)|_{\max}$.

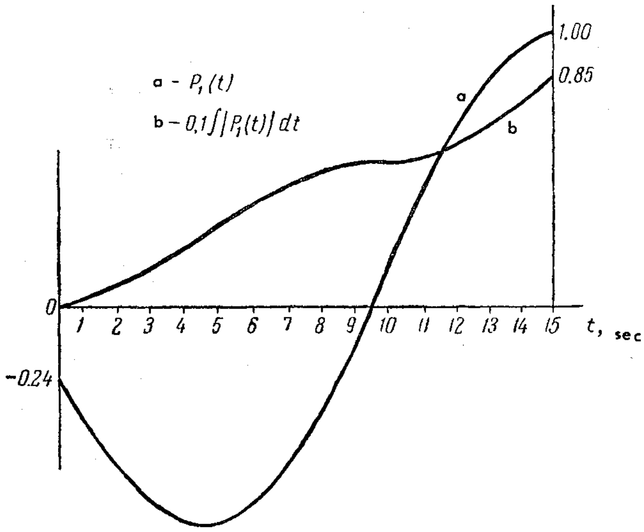

Fig. 1

The adjoint system of equations in this case assumes the form

$$\begin{aligned} \dot{p}_1 &= 0.3p_2 \\ \dot{p}_2 &= -0.3p_1 + 0.04p_2 \end{aligned} \tag{11}$$

To find the initial values $p_1(0)$ and $p_2(0)$, we must run the integration of system (11) "backwards" in time, which is equivalent to a "forward" integration of the following system of equations [see Eq. (8)]:

$$\begin{aligned} \dot{p}_1(\tau) &= -0.3p_2(\tau) \\ \dot{p}_2(\tau) &= 0.3p_1(\tau) - 0.04p_2(\tau) \end{aligned} \tag{12}$$

and here $p_1(\tau)|_{\tau=0} = 1$ and $p_2(\tau)|_{\tau=0} = 0$ [see Eq. (9)].

During the integration of system (12), the analog computer is used to find the values of the variables at time $T = 15$ sec:

$$p_1(\tau)|_{\tau=T} = -0.24 \qquad p_2(\tau)|_{\tau=T} = -0.72$$

The values obtained here are the initial values sought for variables $p_1(t)$ and $p_2(t)$ in the adjoint system (11).

Integration of the adjoint system (11) using initial values $p_1(0) = -0.24$, $p_2(0) = -0.72$ enables us to find the values of $p_1(t)$ and $K_1\int_0^T|p_1(t)|dt$ (see the oscillograms in Fig. 1). The magnitude of the cumulative deviation is

$$|x_1(T)|_{\max} = 0.85$$

As already stated, the maximum deviation in the dynamical system is due to a change in the external disturbances obeying the law,[7] which in our case is of the form

$$f_1(t) = 0.1 \operatorname{sign} p_1(t)$$

As an illustration of this statement, the fundamental system of equations (10) was simulated by analog techniques for various forms of the function $f_1(t)$. (However, in all cases the constraint $|f_1(t)| \leqslant 0.1$ was observed.) In Fig. 2, we see the oscillogram of the transient with respect to coordinate x_1 at $f_1(t) = 0.1 \operatorname{sign} p_1(t)$; $p_1(0) = -0.24$; $p_2(0) = -0.72$. Figure 3 presents oscillograms of transients in the system with respect to coordinate x_1, given the same law governing the variation of $f_1(t)$ and the following initial conditions for $p_1(t)$ and $p_2(t)$:

$p_1(0) = -0.24$	$p_2(0) = -0.72$	curve 1
$p_1(0) = -0.25$	$p_2(0) = -0.25$	curve 2
$p_1(0) = 0$	$p_2(0) = -0.75$	curve 3

The oscillograms demonstrate that the maximum cumulative deviation is obtained only for certain initial conditions applying to $p_i(t)$ and satisfying condition (5) (curve 1). The behavior of the system (with respect to coordinate x_1) in response to other laws of variation of $f_1(t)$ is shown on the oscillograms in Fig. 3, for which $f_1(t) = 0.1 \sin 2\pi\nu t$, and curve 4 corresponds to $\nu = 0.05$ cps; curve 5 corresponds to $\nu = 0.1$ cps, and curve 6 to $\nu = 0.2$ cps. As the frequency increases, the magnitude of the cumulative disturbance

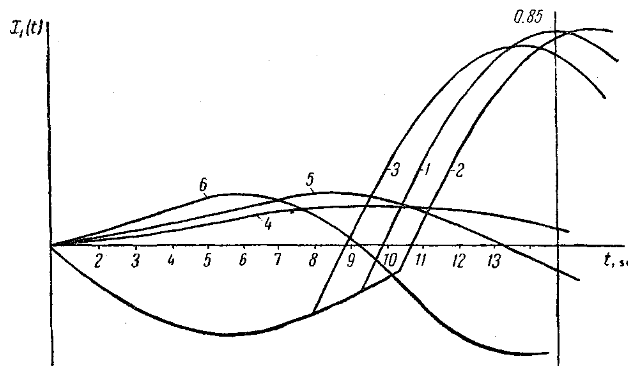

Fig. 3

starts to decline. It is clear from inspection of the curves that the maximum deviation in the case of a harmonic disturbance does not exceed the value 0.4, whereas the maximum cumulative deviation is 0.85.

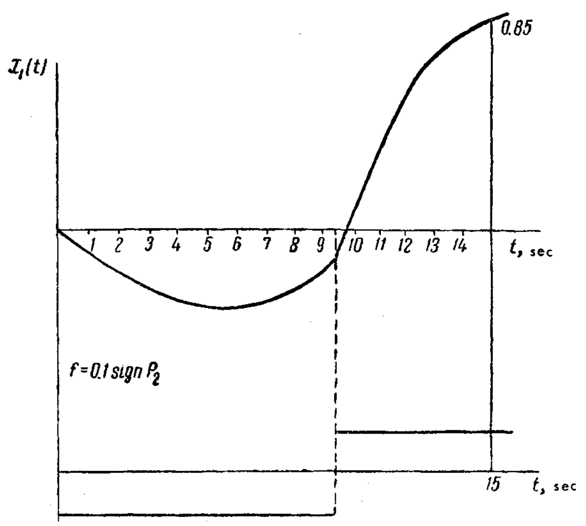

Fig. 2

Analog Solution of the "Hit" Problem in Linear Systems

The problem of cumulative disturbances is closely connected with the "hit" problem. This is the problem of finding the law of variation of forces acting on a dynamical system, in response to which several coordinates of the system (termed controlled coordinates in the following) assume predetermined values at a predetermined instant of time. This problem has been studied in detail, in its most general formulation, by Roitenberg.[6]

Consider now a dynamical system described by the equations of motion (1). We restrict our inquiry to the case of random inputs (forces) which maintain a constant value over finite time intervals and pose the problem of finding the value of those inputs at which the conditions

$$x_m(T) = \bar{x}_m \qquad m = 1, \ldots, r$$

where $\bar{x}_m$ are preassigned numbers are fulfilled.

For the solution of this problem, we turn to Eq. (3). We introduce the terminology $p_i(t)_m$, $i = 1, \ldots, n$ for functions satisfying the adjoint system (2) and the following initial conditions:

$$[p_i(T)]_m = 1 \quad \text{for} \quad i = m$$
$$[p_i(T)]_m = 0 \quad \text{for} \quad i \neq m$$
$$i = 1, \ldots, n \qquad m = 1, \ldots, r$$

Equation (3) may be transformed for $t = T$:

$$\sum_{i=1}^{n} \int_0^T [p_i(\tau)]_m f_i(\tau) d\tau = x_m(T) - \sum_{i=1}^{n} x_i(0) [p_i(0)]_m \qquad m = 1, \ldots, r \tag{13}$$

These are the equations from which we must determine the input disturbances $f_i(t)$.

The initial values $[p_i(0)]_m$ entering here must be determined by the method outlined in the preceding section. We introduce the notation

$$N_m = x_m(T) - \sum_{i=1}^{n} \dot{x}_i(0) [p_i(0)]_m \qquad m = 1, \ldots, r \tag{14}$$

After the values of $[p_i(0)]_m$ have been determined, N_m values will be known numbers.

Accordingly, we obtain from Eq. (13) r equations useful in determining the external disturbances f_i:

$$\sum_{i=1}^{n} \int_0^T [p_i(\tau)]_m f_i(\tau) d\tau = N_m \qquad m = 1, \ldots, r \tag{15}$$

We first consider the case in which all the system coordinates are controlled coordinates ($r = n$) and the number of external input disturbances (external forces) is equal to the number of controlled coordinates.

Since we are looking for disturbances f_i in the class of sectionally constant functions, we may assume

$$f_i(t) = K_i = \text{const} \qquad 0 \leqslant t \leqslant T$$

and Eqs. (15) transform in this case to a system of linear inhomogeneous algebraic equations for K_i:

$$c_{11}K_1 + \ldots + c_{1n}K_n = N_1$$
$$c_{n1}K_1 + \ldots + c_{nn}K_n = N_n$$

where

$$c_{mi} = \int_0^T [p_i(\tau)]_m d\tau$$

For the case in which not all the coordinates in the system are controlled coordinates ($r < n$) but in which the number of external disturbances (forces) is, as before, equal to the number of controlled coordinates, we again obtain a system of algebraic equations whose order will be r.

The case where the number of input disturbances (external forces) is less than the number of controlled coordinates is of particular interest. Since in this case the number of unknown quantities (inputs) will be less than the number of equations which these disturbances must satisfy, the problem has no solution. However, it may be solved by using the approach described in a paper mentioned earlier,[6] in which the author divides interval $(0,T)$ into several equal or unequal subintervals and looks for disturbing step functions $f_i(t)$, which remain constant over the subintervals.

This partition into subintervals makes it possible to increase artificially the number of input disturbances so that it becomes equal to the number of controlled coordinates. For example, let a disturbing force $f_j(t)$ be acting on a system, and let it be required to determine the function $f_j(t)$ (from the class of step functions) satisfying the conditions $x_m(T) = \bar{x}_m (m = 1, \ldots, r)$. Following the idea developed in Ref. 6, we partition the open interval $(0,t)$ into r subintervals $(0,t_1)$, $\ldots$, (t_{r-1},T). We now denote as $K_{j1}, \ldots, K_{jr}$ the values of the step function $f_j(t)$ on each subinterval. Then system of equations (15) may be recast in the following form:

$$d_{11}K_{j1} + \ldots + d_{1r}K_{jr} = N_1$$
$$d_{r1}K_{j1} + \ldots + d_{rr}K_{jr} = N_r \tag{16}$$

where

$$d_{kl} = \int_{t_{l-1}}^{t_l} [p_j(\tau)]_k d\tau \tag{17}$$

It should be noted that, although the partitioning into subintervals is quite arbitrary, the determinant consisting of coefficients d_{kl} must not vanish.

As an example, let us solve the "hit" problem in the system described by

$$\dot{x}_1 = 0.3x_2 + f_1(t)$$
$$\dot{x}_2 = -0.2x_1 - 0.15x_2 \tag{18}$$
$$x_1(0) = 1 \qquad x_2(0) = 0.1$$

We must find function $f_1(t)$ (in the class of step functions) capable of satisfying conditions*

$$x_1(T) = x_2(T) = 0 \qquad T = 10 \text{ sec}$$

Since the number of disturbing forces here is less than the number of controlled coordinates, interval $(0,T)$ is broken down into two subintervals, denoted as K_{11} and K_{12}, the values of the step function $f_1(t)$ on those subintervals.

In order to find numerical values for N_1 and N_2 [see Eq. (14)], we first must find the initial values (of which there will be two sets in this case) for the functions satisfying the adjoint system of equations.

After performing the "backward" integration of the adjoint system on an analog computer, we obtained numerical values

$$[p_1(0)]_1 = 0.1 \qquad [p_2(0)]_1 = 0$$
$$[p_1(0)]_2 = 0 \qquad [p_2(0)]_2 = 0.1$$

Computations based on Eq. (14) yielded

$$N_1 = -0.1 \qquad N_2 = -0.01$$

Analog computer integration of the adjoint system furnished us with a set of values for d_{kl} [see Eq. (17)]—(here, t_1 is assigned the value of 5 sec):

$$d_{11} = 0.35 \qquad d_{12} = -0.36$$
$$d_{21} = 0.27 \qquad d_{22} = 0.67$$

The system of algebraic equations (16) assumed the form

$$0.35K_{11} - 0.36K_{12} = -0.1$$
$$0.27K_{11} + 0.67K_{12} = -0.01$$

On solving this sytem, we have

$$K_{11} = -0.21 \qquad K_{12} = 0.09$$

To illustrate the "hit" problem under consideration, the initial dynamical system is simulated by an analog computer. Figures 4 and 5 present oscillograms of transients relative to coordinates x_1 and x_2, both in an autonomous system (without disturbing forces) and with the disturbing forces that have been computed, a graph of which appears in Fig. 4.

Solution on Analog Computers of the Problem of a Buildup of Perturbations in Nonlinear Systems

In the solution of the problem of a cumulative buildup of perturbations in nonlinear dynamical systems, it is conven-

* In this case, the "hit" problem coincides with the problem of accelerated reduction of a dynamical system to the equilibrium position.[7]

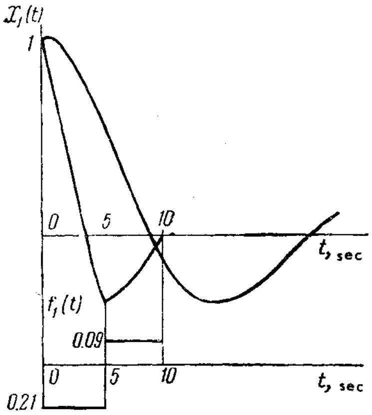

Fig. 4

ient to use the Pontryagin maximum principle, which is widely used in optimization theory.[8, 9]

Let the dynamical system in question be described by the following nonlinear equations:

$$x_i = \varphi_i(x_1, \ldots, x_n, f_1, \ldots, f_m, t) \qquad i = 1, \ldots, n \tag{19}$$

Here, as earlier, $x_1, \ldots, x_n$ are the system coordinates; $f_1, \ldots, f_m$ are the external disturbances, bounded in absolute value.

The problem reduces to a determination of the maximum cumulative deviation in the kth coordinate of the system in the worst-case variant of effect of external forces.

The solution to the problem of cumulative perturbations in a linear system has led us to the necessity of considering the solution of some auxiliary system of equations run on an analog computer. This auxiliary system is the adjoint system. The nature of the transients in the original system is intimately related to the nature of the transients in the adjoint system and conversely. Both systems may be regarded as parts of some overall dynamical system. The symmetrical character of the equations of the original and adjoint systems allow us to regard these equations as the canonical equations of motion of an overall dynamical system.

Extending this idea to the nonlinear case, we supplement the initial dynamical system (19) by some auxiliary system such that the equations of motion of the overall system formed in this manner will have the form of canonical equations in the Hamiltonian form.

It is not difficult to realize that the Hamiltonian that is required must have the form

$$H(x_1, \ldots, x_n p_1, \ldots, p_n f_1, \ldots, f_m t) = \sum_{i=1}^{n} p_i \dot{x}_i - \sum_{i=1}^{n} p_i \varphi_i \tag{20}$$

The canonical equations of motion of the overall system[10] are written:

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i} \qquad i = 1, \ldots, n \tag{21}$$

Consequently, the extension of the original system to the complete system is carried out such that the first group of canonical equations coincides with equations of motion (19) of the original nonlinear dynamical system. The second group of equations (21) then also determines the motion of the auxiliary system, which we shall term an adjoint system by analogy with the linear case. Therefore, functions $p_1(t), \ldots, p_n(t)$ are the coordinates of the adjoint system and as such may also be viewed as momenta in the overall system.

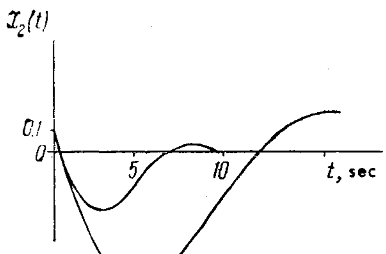

Fig. 5

In optimization theory, the problem of the choice of proper law for disturbing functions $f_i(t)$ optimizing at some fixed instant T the linear form

$$S = \sum_{i=1}^{n} c_i x_i(T)$$

is considered in particular.

It is readily shown that this requires optimization of the Hamiltonian function H with respect to disturbing forces $f_i(t)$. In fact, consider an expression identically vanishing whenever Eqs. (20) and (21) are satisfied:

$$\sum_{i=1}^{n} p_i \dot{x}_i - H \equiv 0$$

If we consider the variation of this expression while taking the variation of the disturbing functions $f_i(t)$, then, by virtue of Eqs. (21), this variation will be zero at each instant of time. Accordingly, the following is valid:

$$\delta \int_0^T \left\{ \sum_{i=1}^{n} \dot{x}_i p_i - H \right\} dt = 0$$

By transforming the latter expression, we obtain

$$\sum_{i=1}^{n} p_i \delta x_i \Big|_0^T - \sum_{i=1}^{n} \int_0^T \dot{p}_i \delta x_i dt + \sum_{i=1}^{n} \int_0^T \dot{x}_i \delta p_i dt - \int_0^T \delta H \, dt = 0 \tag{22}$$

Moreover

$$\delta H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i} \delta x_i + \frac{\partial H}{\partial p_i} \delta p_i \right) + \sum \frac{\partial H}{\partial f_i} \delta f_i$$

Substitution of the value of δH into Eq. (22) eventually yields, with Eqs. (21) taken into account

$$\sum_{i=1}^{n} p_i \delta x_i \Big|_0^T = \int_0^T \sum_{i=1}^{m} \frac{\partial H}{\partial f_i} \delta f_i dt \tag{23}$$

In optimizing the linear form $S(T)$, condition

$$\delta S = \sum_{i=1}^{n} c_i \delta x_i(T) = 0$$

must be satisfied. Comparing this last equation to Eq. (23) and also bearing in mind that $\delta x_i(0) = 0$, we arrive at the conclusion that, provided conditions $p_i(T) = c_i$ and $i = 1, \ldots, n$ are fulfilled, optimization of form $S(T)$ requires optimization (with respect to disturbing functions f_i) of function H.

Detailed analysis with investigation of the sign of the second variation shows that, in order to maximize (minimize) the form S at a fixed instant T, it is necessary to maximize (minimize) function H relative to the input disturbances f_i. This is, in fact, the gist of the Pontryagin maximum principle for the problem in question (9).

Note that, for a linear system described by Eqs. (1), function H assumes the form

$$H = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_k p_i + \sum_{i=1}^{n} p_i f_i$$

and, consequently, in accord with the maximum principle, we arrive at the following necessary law of variation for f_i:

$$f_i(t) = K_i \operatorname{sign} p_i(t) \qquad |f_i(t)| \leqslant K_i$$

The need for such a law of f_i in the problem of maximization of the linear form

$$S = \sum_{i=1}^{n} c_i x_i(T)$$

is particularly apparent when we recall condition (4).

In fact, assuming $c_i = p_i(T)$ in (4), we immediately obtain

$$\sum_{i=1}^{n} c_i x_i(T) \equiv S = \int_0^T \sum_{i=1}^{n} p_i(\tau) f_i(\tau) d\tau$$

which illustrates the need for the foregoing law on variations in $f_i(t)$.

Returning to the problem of cumulative perturbations, note that it flows from the problem of maximization of the linear form S, with the assumption that $c_i = 0$ for $i \neq k$, $c_k = 1$. Consequently, in a nonlinear dynamical system subjected to external influences, the maximum cumulative deviation in the kth coordinate at an instant T will occur in response to such $f_i(t)$ as are maximized at each instant of time with respect to the disturbing inputs of the Hamiltonian function H of the entire system. Functions $p_1(t), \ldots, p_n(t)$ included in the expression for the Hamiltonian H must satisfy the adjoint system and conditions $p_i(T) = 0$ for $i \neq k$, $p_k(T) = 1$. As in the linear case, functions $p_i(t)$ must be realized by a computer simulating the adjoint system. As in the previous case, we encounter the problem of finding the initial values for functions p_i. The difficulty is that, in the nonlinear case, the adjoint system includes the coordinates of the original system, which prevents our using the method of "backwards" integration of the adjoint system. From the mathematical viewpoint, we are dealing here with a system of nonlinear differential equations of order $2n$, where, for n variables (the coordinates of the original system), the initial values are known, and, for the remaining $2n - n$ variables (coordinates of the adjoint system), the final values are known. Finding the variables of the adjoint system from these initial-value data is an independent problem which can be solved, in particular, by means of specialized analog devices.

In the absence of such special-purpose analog computers, this problem may be solved on ordinary analog computers by the search method. For example, consider the nonlinear second-order oscillating system subjected to an external force bounded in absolute value. The equation of the system has the form

$$\ddot{x}_1 + (1 + \gamma x_1^2) x_1 = f(t) \qquad (24)$$

We shall solve the problem of maximum cumulative value of coordinate x_1 at some fixed time T for the following numerical parameters

$$\gamma = 0.1 \qquad |f(t)| \leqslant K = 0.1$$
$$T = 4.2 \text{ sec} \qquad x_1(0) = \dot{x}_1(0) = 0$$

Let us present Eq. (24) in the form of the system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - 0.1x_1^3 + f(t)$$

In accord with Eq. (2), the expression for the Hamiltonian has the form

$$H = x_2 p_1 - x_1 p_2 - 0.1 x_1^3 p_2 + f p_2 \qquad (25)$$

This leads directly to the adjoint system

$$\dot{p}_1 = p_2 + 0.3 x_1^2 p_2$$
$$\dot{p}_2 = -p_1 \qquad (26)$$

On the basis of the foregoing discussion and also from the form of function H, Eq. (25), we infer that the maximum deviation of coordinate x_1 at the instant of time T will be, in obedience to the following law of variation of the force:

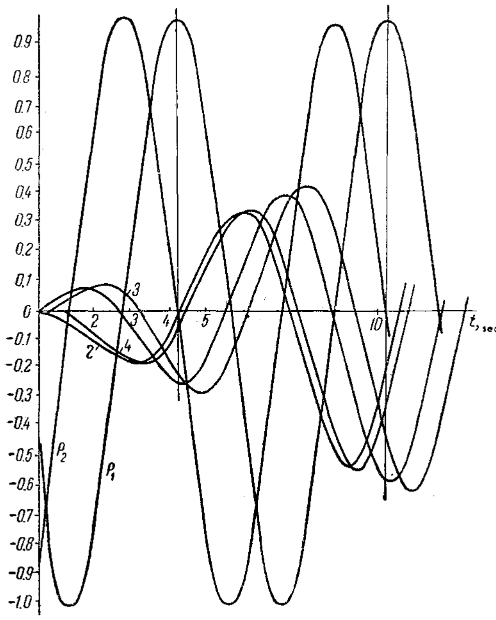

Fig. 6

$$f(t) = K \operatorname{sign} p_2(t)$$

Function $p_2(t)$ appearing here is found from the solution of system (26), which we seek under the following conditions:

$$p_1(T) = 1 \qquad p_2(T) = 0 \qquad (27)$$

The search method (sequential scanning of all possible values) was used to find the initial values of the functions $p_1(0)$ and $p_2(0)$ satisfying the condition (27). These were found to be

$$p_1(0) = -0.38 \qquad p_2(0) = -0.93$$

As an illustration of the foregoing, we reproduce oscillograms of the transients in the original system and in the adjoint system, both analog-simulated.

Figure 6 shows oscillograms of functions $p_1(t)$ and $p_2(t)$ as solutions of the adjoint system at $p_1(0) = -0.38$, $p_2(0) = -0.93$. Moreover, the diagram displays oscillograms of transients of this dynamical system with respect to coordinate x_1 for the input disturbance $f = 0.1 \operatorname{sign} p_2$ at various initial values $p_1(0)$ and $p_2(0)$.

Curve 1 corresponds to the dynamical process at $p_1(0) = -0.38$; $p_2(0) = -0.93$. Curve 2 corresponds to the dynamical process at $p_1(0) = -0.38$; $p_2(0) = 0$. Curve 3 corresponds to the dynamical process at $p_1(0) = 0$; $p_2(0) = -0.93$. Curve 4 corresponds to the dynamical process at $p_1(0) = -0.75$; $p_2(0) = -0.2$. A comparison of curves 1 to 4 demonstrates that the maximum deviation in coordinate x_1 is to be observed on curve 1, corresponding to the initial conditions satisfying (27). The value of the maximum accumulated disturbance is then found to be 0.22.

Oscillograms of the transients in the same coordinate x_1 in response to a harmonic law of variation in the input disturbance $f = 0.1 \sin 2\pi\nu t$ enable us to determine the maximum deviation in coordinate x_1, namely 0.21, that is, less than in the worst-case law of variation for an external disturbance.

Summary

The feasibility of employing analog computers to solve such problems as cumulative disturbances, "hits," and problems involving high speed equilibration of dynamical systems, was demonstrated previously. In the last two cases, the electronic analog functions as an inseparable part of the automatic control system, making it possible to shape the required inputs.

The author expresses his gratitude to R. A. Velershtein and N. V. Prolygina, his co-workers in the Oscillations Laboratory, who rendered valuable assistance in the completion of the experiment.

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Reviewer's Comment

This paper describes an analog computer technique for systems analysis for determining a "worst worst-case." The reviewer is not aware of the application of this technique in this country, although alternate methods are employed to obtain equivalent design information. Therefore, the paper is a useful addition to the literature.

The author employs a set of "adjoint equations" to obtain intermediate results. It should be noted that his use of the term "adjoint" differs from the use of the same term in this country. As applied by Laning and Battin[1] and others,[2–4] the "adjoint method" is used for determining weighting functions for time-varying linear systems. The latter are closely related to the rms response of a system to random inputs.

One distinguishing difference of Savinov's development is his emphasis on worst-case performance. In this, he is close to some of the techniques used in analysis for optimum control systems (cf. Chang[5]). The example for the "hit" problem resembles a contactor or "bang-bang" servo system. The extension to nonlinear systems is of particular interest. Conceivably these computer solutions might be implemented in fast-time to permit on-line optimization of a system or process.

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Reliability Computation of Complex Automated Systems

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Computing the reliability of a system means determining its quantitative reliability parameters from the known parameters of the elements which make up the system.

A complete description of reliability can be given in terms of the probability of operation without failure $P(t)$, average time of reliable operation T_{av}, and the likelihood of failure $\lambda(t)$. The most widely used methods of computing the foregoing parameters are based on the assumption that the failures have a Poisson distribution and that an exponential reliability law applies.[1] In this case, to compute $P(t)$, T_{av}, and $\lambda(t)$ of the system, the likelihood of failure must be known for all elements of the basic system. Since the likelihood of failure of the elements depends essentially on their schedules and usage, one must have a family of curves which will determine the likelihood of failure as a function of the loading factors, the temperature of the ambient medium, the amplitude and frequency of vibrations, humidity, etc. Such relations are not available at present for most elements of automated systems. Further, the forementioned computational methods cannot be applied to the calculation of quantitative reliability parameters under various conditions of usage of the same system and make the formulation of reliability criteria difficult for separate parts and units of a complex system, and also lead to great computational errors. The coefficient method given here for computing the reliability of complex systems is free, to some extent, of the forementioned difficulties.

The method is based on the following assumptions: The failures are random and independent events; the failure of any element leads to failure of the entire system; the likeli-

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